The rational sectional category of certain maps

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To my mother

Abstract

We give a simple algebraic characterisation of the sectional category of rational maps admitting a homotopy retraction. As a particular case we get the Felix-Halperin theorem for rational Lusternik-Schnirelmann category and prove the conjecture of Jessup-Murillo-Parent on rational topological complexity. We also give a characterisation for relative category in the sense of Doeraene-El Haouari.

2010 *Mathematics Subject Classification* : 55M30, 55P62.
Keywords: Rational homotopy, sectional category, topological complexity.

Introduction

Throughout this work we consider all spaces to be of the homotopy type of simply connected CW-complexes of finite type and use standard rational homotopy techniques which are explained in the excellent text [13]. Sectional category is an invariant of the homotopy type of maps introduced by Schwarz in [21]. If $f\colon X \to Y$ is a continuous map, its *sectional category* is the smallest m for which there are $m+1$ local homotopy sections for f whose sources form an open cover of Y. Its most studied particular case is the well known Lusternik-Schnirelmann (LS) category of a space X introduced in [18] as a lower bound for the number of critical points on any smooth map defined

This work was partially supported by FEDER through the Ministerio de Educación y Ciencia project MTM2013-41768-p and the Belgian Interuniversity Attraction Pole (IAP) within the framework “Dynamics, Geometry and Statistical Physics” (DYGEST P7/18)

on a smooth manifold X. Namely, the LS category of a pointed space X, $\mathrm{cat}(X)$, is the sectional category of the base point inclusion map, $* \hookrightarrow X$.

A remarkable theorem of Félix-Halperin [12, Theorem 4.7] gives an algebraic characterisation of LS category of rational spaces in terms of their Sullivan models. Explicitly, if X is a space modelled by $(\Lambda V, d)$ and X_0 is its rationalisation (see [13, 23]) then $\mathrm{cat}(X_0)$ is the smallest m for which the commutative differential graded algebra (cdga) projection

$$\rho_m\colon\ (\Lambda V, d) \to \left(\frac{\Lambda V}{\Lambda^{>m} V}, \overline{d}\right)$$

admits a homotopy retraction, that is, a strict retraction for a relative Sullivan model for ρ_m.

Let $f\colon X \to Y$ be a map such that its rationalisation f_0 admits a homotopy retraction r. Then, through standard rational homotopy techniques f can be modelled by a retraction $\varphi\colon (B \otimes \Lambda W, D) \to (B, d)$ of a relative Sullivan algebra $(B, d) \rightarrowtail (B \otimes \Lambda W, D)$ modeling r. For simplicity in the notation, write $\varphi\colon A \to B$ and call it from now on an *s-model* of f.

Theorem 1. *The sectional category of the rationalisation of f,* $\mathrm{secat}(f_0)$*, is the smallest m for which the cdga projection*

$$A \to \frac{A}{(\ker \varphi)^{m+1}}$$

admits a homotopy retraction.

Observe that, choosing $\varphi\colon (\Lambda V, d) \to \mathbb{Q}$, this theorem reduces to the Félix-Halperin theorem for rational LS-category. On the other hand, it also generalises the Murillo-Jessup-Parent conjecture on rational topological complexity [16].

Indeed, in his famous paper [9] M. Farber introduced the concept of *topological complexity* of a space X, $\mathrm{TC}(X)$, which can be seen as the sectional category of the diagonal map $\Delta\colon X \to X \times X$. This invariant is used to estimate the *motion planning complexity* of a mechanical system and also has applications to other fields of mathematics [10]. As a direct generalisation of this invariant, Rudyak introduced in [20] the concept of higher topological

n-complexity of a space, $\mathrm{TC}_n(X)$, as the sectional category of the n-diagonal map $\Delta_n\colon X \to X^n$. Several explicit computations of topological complexity of rational spaces have been done in [2, 15, 16, 17]. Inspired on the Félix-Halperin theorem, Jessup, Murillo and Parent, conjectured that $\mathrm{TC}(X_0)$ is the smallest m such that the projection

$$(\Lambda V \otimes \Lambda V, d) \longrightarrow \left(\frac{\Lambda V \otimes \Lambda V}{K^{m+1}}, \overline{d}\right)$$

admits a homotopy retraction, where K denotes the kernel of the multiplication morphism $\mu_2\colon \Lambda V \otimes \Lambda V \to \Lambda V$.

Theorem 1 applied to higher topological complexity is a bit more general than the Murillo-Jessup-Parent conjecture. Namely, if A is *any* cdga model for a space X, then Δ_n admits an s-model of the form $\varphi = (\mathrm{Id}_A, \eta, \dots, \eta)\colon A\otimes (\Lambda V)^{\otimes n-1} \to A$ where $\eta\colon \Lambda V \xrightarrow{\simeq} A$ is a Sullivan model for A. From Theorem 1 we immediately deduce:

Theorem 2. *Let X be a topological space, then $\mathrm{TC}_n(X_0)$ is the smallest m such that the projection*

$$A \otimes (\Lambda V)^{\otimes n-1} \longrightarrow \frac{A \otimes (\Lambda V)^{\otimes n-1}}{(\ker\ \varphi)^{m+1}}$$

admits a homotopy retraction.

We remark that, since $\mathrm{secat}(f_0) \leq \mathrm{secat}(f)$ [2], we get algebraic lower bounds for *integral* sectional category which are better than $\operatorname{nil}\ker f^*$. Some of the ideas in this paper come from [4].

1 Fibrewise pointed cdgas and relative nilpotency

In this section we develop some technical tools that will be needed later on. Let $\mathcal{C}$ be a J-category in the sense of Doeraene [5, 6] or a closed model category in the sense of Quillen [19] and fix and object B of $\mathcal{C}$. Consider the *fibrewise pointed category* over B [1, Pg. 30], denoted by $\mathcal{C}(B)$, whose objects are factorisations of Id_B, $B \xrightarrow{s_X} X \xrightarrow{p_X} B$, and whose morphism are

morphisms in $\mathcal{C}$, $f\colon X \to Y$, such that $f \circ s_X = s_Y$ and $p_Y \circ f = p_X$. Such a morphism is said to be a *fibration* ($\twoheadrightarrow$), *cofibration* ($\rightarrowtail$) or *weak equivalence* ($\xrightarrow{\simeq}$) if the underlying morphism f is such in $\mathcal{C}$. With these definitions $\mathcal{C}(B)$ is also either a J-category or a closed model category (note that this structure is not the same as that of [14]). We denote by $[X, Y]_B$ the homotopy classes of morphism in $\mathcal{C}(B)$ from the fibrant-cofibrant object X into Y.

Now, and for the rest of the paper, we particularise on $\mathcal{C} = \mathbf{cdga}$. Remark that the fibrant-cofibrant objects of $\mathbf{cdga}(B)$ are precisely the relative Sullivan algebras $(B \otimes \Lambda W, D)$ with the natural inclusion $B \hookrightarrow (B \otimes \Lambda W, D)$ and endowed with a given retraction. In this context, the general property [19] by which weak equivalences induce bijections on homotopy classes reads:

Lemma 3. *Suppose $\theta\colon A \to C$ is a quasi-isomorphism in $\mathbf{cdga}(B)$ and $(B \otimes \Lambda V, D)$ a fibrant-cofibrant object of $\mathbf{cdga}(B)$, then composition with θ induces a bijection $\theta_\#\colon [B \otimes \Lambda V, A]_B \to [B \otimes \Lambda V, C]_B$.*

Definition 4. *Let $A \in \mathbf{cdga}(B)$, its* relative nilpotency index*, $\mathrm{nil}_B(A)$, is the nilpotency index* $\mathrm{nil}\ker p_A$ *of the ideal* $\ker p_A$.

The following lemma is crucial. It tells us that we can control the relative nilpotency index of certain homotopy pullbacks of $\mathbf{cdga}(B)$.

Lemma 5. *Let $i\colon C \rightarrowtail (C \otimes \Lambda V, D)$ be a cofibration in $\mathbf{cdga}(B)$ such that $D(V) \subset (\ker p_C) \oplus (C \otimes \Lambda^+ V)$ and $p_{C\otimes\Lambda V}(V) = 0$. Then, there is an object $N \in \mathbf{cdga}(B)$ weakly equivalent to the homotopy pullback of i and $s_{C\otimes\Lambda V}$ for which $\mathrm{nil}_B\ N = \mathrm{nil}_B\ C + 1$.*

Proof. In $\mathbf{cdga}(B)$, factor $s_{C\otimes\Lambda V}$ as $B \overset{\alpha}{\hookrightarrow} S \overset{h}{\twoheadrightarrow} C \otimes \Lambda V$ where

$$S = B \oplus \left(C \otimes \Lambda V \otimes \Lambda^+(t, dt)\right),$$

in which t has degree 0, $b(c \otimes v \otimes \xi) = s_C(b) c \otimes v \otimes \xi$, and $h(c \otimes v \otimes t) = c \otimes v$. As $C \otimes \Lambda V \otimes \Lambda^+(t, dt)$ is acyclic, α is a quasi-isomorphism and thus, the homotopy pullback of i and $s_{C\otimes\Lambda V}$ is the pullback

$$\begin{tikzcd} M' \arrow[r, "\overline{h}"] \arrow[d] & C \arrow[d, tail, "i"] \\ S \arrow[r, two heads, "h"'] & C \otimes \Lambda V \end{tikzcd}$$

of i and h. This is in fact a pullback in $\mathbf{cdga}(B)$ by choosing $p_{M'} = p_C \circ \overline{h}$ and $s_{M'} = (\alpha, s_C)$. To finish, we will construct an object N of $\mathbf{cdga}(B)$ weakly equivalent to M' with $\operatorname{nil}_B N = \operatorname{nil}_B C + 1$.

Write $K_\epsilon = \ker \epsilon$ where $\epsilon\colon \Lambda^+(t, dt) \to \mathbb{Q}$ is the augmentation sending t to 1, and consider the $\mathbf{cdga}(B)$ isomorphism

$$\eta\colon M \xrightarrow{\cong} M',$$

in which:

$$M = B \oplus \big(C \otimes \Lambda^+(t, dt)\big) \oplus \big(C \otimes \Lambda^+ V \otimes K_\epsilon\big)\,,$$

$$s_M(b) = b, \quad p_M(b) = b, \quad p_M(c \otimes \xi) = p_C(c)\epsilon(\xi), \quad p_M(c \otimes v \otimes \omega) = 0,$$

$$\eta(b) = (b, s_C(b)), \quad \eta(c\otimes\xi) = (c\otimes 1\otimes\xi, c\epsilon(\xi)), \quad \eta(c\otimes v\otimes\omega) = (c\otimes v\otimes\omega, 0),$$

with $b \in B$, $c \in C$, $\xi \in \Lambda^+(t, dt)$, $v \in V$ and $\omega \in K_\epsilon$.

Next, write $C = \ker p_C \oplus R$ and consider

$$N = B \oplus \big(\ker p_C \otimes \Lambda^+(t, dt)\big) \oplus \big(\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon\big) \oplus \big(R \otimes \Lambda^+ V \otimes dt\big)$$

which, since $D(V) \subset (\ker p_C) \oplus (C \otimes \Lambda^+ V)$, is a sub $\mathbf{cdga}(B)$ of M. Moreover, the inclusion $N \hookrightarrow M$ is a weak equivalence in $\mathbf{cdga}(B)$ as the subcomplexes $(\ker p_C \otimes \Lambda^+(t, dt)) \oplus (\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon)$ and $(C \otimes \Lambda^+(t, dt)) \oplus (\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon)$ are quasi-isomorphic and the inclusion of quotient complexes $B \oplus (R \otimes \Lambda^+ V \otimes dt) \hookrightarrow B \oplus (R \otimes \Lambda^+ V \otimes K_\epsilon)$ is a quasi-isomorphism.

Finally, we have that

$$\ker p_N = \big(\ker p_C \otimes \Lambda^+(t, dt)\big) \oplus \big(\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon\big) \oplus \big(R \otimes \Lambda^+ V \otimes dt\big)\,,$$

and thus, a non-trivial product of maximal length in this ideal is given by $z(1 \otimes v \otimes dt)$ where z is a non-trivial product of maximal length in $\ker p_C \otimes \Lambda^+(t)$. This proves that $\operatorname{nil}_B N = \operatorname{nil}_B C + 1$. $\square$

2 The main result

Let $f\colon X \to Y$ be a continuous map. Recall from [1, 3, 11] that, by iterated joins, one can construct an m-Ganea map for f, $G_m(f)$, fitting into a

commutative diagram

$$
\begin{tikzcd}
& X \arrow[dl, "\iota"'] \arrow[dr, "f"] & \\
P^m(f) \arrow[rr, "G_m(f)"'] & & Y,
\end{tikzcd}
\tag{1}
$$

and that $\operatorname{secat}(f) \leq m$ if and only if $G_m(f)$ admits a homotopy section. Also, if $\varphi\colon A \twoheadrightarrow B$ is a surjective model for f, then Diagram (1) can be modelled by a diagram

$$
\begin{tikzcd}
A \arrow[rr, "\kappa_m"] \arrow[dr, two heads, "\varphi"'] & & C_m \arrow[dl, two heads, "p_m"] \\
& B, &
\end{tikzcd}
$$

where κ_m models $G_m(f)$ and can be constructed inductively by taking the homotopy pullback of the induced maps by the homotopy pushout of φ and any model, $g\colon A \to D$, of $G_{m-1}(f)$. Standard arguments show that $\operatorname{secat}(f_0) \leq m$ if and only if κ_m admits a homotopy retraction. One can extend this to:

Definition 6. *Let $f\colon X \to Y$ be a continuous map. Then:*

(i) $\operatorname{msecat}(f) \leq m$ *if and only if κ_m admits a homotopy retraction as A-module;*

(ii) $\operatorname{Hsecat}(f) \leq m$ *if and only if κ_m is homology injective.*

We now give the key model for the m-Ganea map $G_m(f)$:

Proposition 7. *Let f be a map such that f_0 admits a homotopy retraction and let $\varphi\colon A \twoheadrightarrow B$ be an s-model for f. Then there is a model λ_m for $G_m(f)$ which is a morphism in* $\mathbf{cdga}(B)$*,*

$$
\begin{tikzcd}
& B \arrow[dl, tail, "s"'] \arrow[dr, "s_m"] & \\
A \arrow[rr, "\lambda_m"] \arrow[dr, two heads, "\varphi"'] & & C_m \arrow[dl, two heads, "p_m"] \\
& B, &
\end{tikzcd}
$$

with $\operatorname{nil}_B C_m = m$*.*

Proof. We will proceed by induction. For $m = 0$ the assertion holds since $p_0 = \mathrm{Id}_B$. Suppose λ_{m-1} exists. Since φ is surjective, one can take a relative Sullivan model for φ, $\theta\colon (A \otimes \Lambda V, D) \xrightarrow{\simeq} B$, such that $D(V) \subset (\ker \varphi) \oplus (A \otimes \Lambda^+ V)$ and $\theta(V) = 0$. Now take the homotopy pushout

$$\begin{tikzcd}
A \arrow[r, "\lambda_{m-1}"] \arrow[d, tail, "j"'] & C_{m-1} \arrow[d, tail, "i"] \arrow[ddr, bend left, "p_{m-1}"] & \\
A \otimes \Lambda V \arrow[r, "\lambda_{n-1} \otimes \mathrm{Id}"'] \arrow[drr, bend right, "\theta"'] & C_{m-1} \otimes \Lambda V \arrow[dr, dashed, "(\theta, p_{m-1})"] & \\
 & & B
\end{tikzcd}$$

and factor $\lambda_{n-1} \otimes \mathrm{Id}$ as $q \circ w$ with $q\colon E \twoheadrightarrow C_{m-1} \otimes \Lambda V$ a surjective cdga morphism and $w\colon A \otimes \Lambda V \xrightarrow{\simeq} E$ a weak equivalence. Then the pullback's universal property

$$\begin{tikzcd}
A \arrow[dr, dashed, "g"] \arrow[drr, bend left, "\lambda_{m-1}"] \arrow[ddr, bend right, "w \circ j"'] & & \\
 & T \arrow[r, "\overline{q}"] \arrow[d] & C_{m-1} \arrow[d, tail, "i"] \\
 & E \arrow[r, two heads, "q"'] & C_{m-1} \otimes \Lambda V,
\end{tikzcd}$$

gives a model g for $G_m(f)$ which can be seen as a morphism in $\mathbf{cdga}(B)$ by taking $p_T = p_{m-1} \circ \overline{q}$ and $s_T = g \circ s$. Now, define $\beta = i \circ s_{m-1}\colon B \to C_{m-1} \otimes \Lambda V$ and consider the factorisation of $\beta = q \circ (w \circ j \circ s)$ as a quasi-isomorphism followed by a fibration. On the other hand, consider also the factorisation of $\beta = h \circ \alpha$ as in the proof of Lemma 5. Applying [6, Lemma 1.8] to previous factorisations and the following commutative square in $\mathbf{cdga}$(B),

$$\begin{tikzcd}
B \arrow[r, "s_{m-1}"] \arrow[d, "\mathrm{Id}_B"'] & C_{m-1} \arrow[d, tail, "i"] \\
B \arrow[r, "\beta"'] & C_{m-1} \otimes \Lambda V,
\end{tikzcd}$$

we get quasi-isomorphisms $M' \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} T$. Now, applying Lemma 5 to $C_{m-1} \rightarrowtail C_{m-1} \otimes \Lambda V$, with $s_{C_{m-1} \otimes \Lambda V} = \beta$ and $p_{C_{m-1} \otimes \Lambda V} = (\theta, p_{m-1})$ we get

an object C_m of $\mathbf{cdga}(B)$, with $\mathrm{nil}_B\ C_m = m$, which is weakly equivalent M'. Observe that we cannot use the pullback's universal property to get a model of $G_m(f)$ because, in general, $\beta \circ \varphi$ does not coincide with $i \circ \lambda_{m-1}$. We get then a diagram in $\mathbf{cdga}(B)$

$$A \xrightarrow{g} T \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} M' \xleftarrow{\simeq} C_m.$$

Since A is a fibrant-cofibrant object of $\mathbf{cdga}(B)$, we can apply Lemma 3 to get a model for $G_m(f)$ in $\mathbf{cdga}(B)$, $\lambda_m \colon A \to C_m$, with $\mathrm{nil}_B\ C_m = m$. $\square$

Given $\varphi \colon A \twoheadrightarrow B$ a surjective cdga morphism, consider, for each $m \geq 0$, the cdga projection

$$\rho_m \colon A \to \frac{A}{(\ker \varphi)^{m+1}}.$$

Then Theorem 1 is just statement *(i)* in the following:

Theorem 8. *Let $\varphi \colon A \twoheadrightarrow B$ be an s-model for a map f such that f_0 admits a homotopy retraction. Then:*

(i) $\mathrm{secat}(f_0)$ *is the smallest m for which ρ_m admits a homotopy retraction;*

(ii) $\mathrm{msecat}(f_0)$ *is the smallest m for which ρ_m admits a homotopy retraction as A-module;*

(iii) $\mathrm{Hsecat}(f)$ *is the smallest m such that $H(\rho_m)$ is injective.*

Proof. Take from Proposition 7 a morphism of $\mathbf{cdga}(B)$, $\lambda_m \colon A \to C_m$, modelling $G_m(f)$ with $\mathrm{nil}_B\ C_m = m$. Since $\lambda_m((\ker\ \varphi)^{m+1}) = 0$ we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\lambda_m"] \arrow[d, "\rho_m"'] & C_m \arrow[d, "p_m"] \\ \frac{A}{(\ker\ \varphi)^{m+1}} \arrow[ur, "\overline{\lambda_m}"'] \arrow[r, "\overline{\varphi}"'] & B \end{tikzcd}$$

and the result follows by standard rational homotopy techniques and [2, Proposition 12]. $\square$

Observe that [2, Example 10] shows that the hypothesis s is a cofibration is necessary.

In [22] D. Stanley gives an example of a map f for which f_0 does not admit a homotopy retraction and $\mathrm{msecat}(f) < \mathrm{secat}(f_0)$. Here we state:

Conjecture 9. *If f is a map and f_0 admits a homotopy retraction, then*

$$\mathrm{msecat}(f) = \mathrm{secat}(f_0).$$

Concerning the rational topological complexity of a given space X and, with the notation in Theorem 2, we may define $\mathrm{mTC}(X)$ as the smallest integer m for which the projection

$$A \otimes \Lambda V \to \frac{A \otimes \Lambda V}{(\ker \varphi)^{m+1}}$$

admits a homotopy retraction as $A \otimes \Lambda V$-module. Then Theorem 8 (ii) combined with [16, Theorem 1.6] gives the Ganea conjecture for mTC.

Theorem 10. *Given any space X then* $\mathrm{mTC}(X \times S^n) = \mathrm{mTC}(X) + \mathrm{mTC}(S^n)$.

We finish by presenting, via Theorem 8, an algebraic description of the rational relative category. Recall [7] that the relative category, $\mathrm{relcat} f$, of a map f is the smallest m for which $G_m(f)$ of Diagram (1) admits a homotopy section s such that $s \circ f \simeq \iota$. Also, in [7], Doeraene and El Haouari proved that $\mathrm{secat}(f)$ and $\mathrm{relcat}(f)$ differ at most by one and conjectured in [8] that they agree on maps admitting a homotopy retraction. Consider then such a map f and $\varphi\colon A \to B$ and s-model for f. This gives a diagram

$$\begin{tikzcd}
 & (A \otimes \Lambda Z_m, D) \arrow[dr, "\theta_m", "\simeq"'] & \\
A \arrow[ur, "i_m"] \arrow[rr, "\rho_m"] \arrow[dr, "\varphi"'] & & \frac{A}{(\ker\ \varphi)^{m+1}} \arrow[dl, "\overline{\varphi}"] \\
 & B, &
\end{tikzcd}$$

where i_m is a relative Sullivan model for ρ_m.

Theorem 11. *With previous notation,* $\mathrm{relcat}(f_0)$ *is the smallest m such that i_m admits a retraction r verifying $\varphi \circ r \simeq \overline{\varphi} \circ \theta_m$ rel A.*

Proof. Consider the commutative diagram in the proof of Theorem 8, where p_m is a model for ι in Diagram (1). Taking j_m a relative model of λ_m and

applying Lemma 3 we get a diagram in $\mathbf{cdga}(B)$

$$\begin{tikzcd} & (A \otimes \Lambda Z_m, D) \arrow[dr, "w"] & \\ A \arrow[ur, tail, "i_m"] \arrow[rr, tail, "j_m"'] & & (A \otimes \Lambda W_m, D)\,. \end{tikzcd}$$

If j_m admits a retraction r' such that $\varphi \circ r' \simeq p_m$ rel A then i_m admits a retraction $r := r' \circ w$ such that $\varphi \circ r = \varphi \circ r' \circ w \simeq p_m \circ \omega = \overline{\varphi} \circ \theta_m$ rel A. $\square$

Acknowledgements

The author is grateful to the referee for his or her helpful remarks. The author also acknowledges the Belgian Interuniversity Attraction Pole (IAP) for support within the framework "Dynamics, Geometry and Statistical Physics" (DYGEST).

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